# *Income distribution, effective demand and limit cycles*†

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# *Income distribution, effective demand and limit cycles*



*Abstract*

In this paper we propose a theoretical model where profit maximising firms bargain with local unions on wages. Relative bargaining power depends on aggregate labour market conditions. Investment decisions can be profit constrained: in such a case a wage increase can reduce investment whenever it lowers the cash flow for the firm. However, given different propensities to consumption of workers and capitalists, a shift in favour of wages in income distribution may compensate profit reduction (due to cost increase) through a demand increase. After studying the dynamic properties of a system of two non-linear differential equations, we use reasonable parameter values (taken from empirical works on the Italian economy) to corroborate our analytical findings.

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## 1. *Introduction*

The pioneering work of R. Goodwin on "A growth cycle" originally published in 1965 [see Goodwin 1969] gives an explanation of the business cycle in terms of an interaction between income distribution and accumulation activity. It has been followed by a massive research trying to extend the original model in several directions: price formation, role of effective demand, labour/capital substitutability, to quote the most relevant ones.

The original model owes a good part of its success to its clear-cut assumptions. Goodwin follows the Cambridge tradition in assuming a constant capital/output ratio: thus when the capital/output ratio does not increase (i.e. the marginal productivity of capital remain constant), the rate of profit does not vary during an expansionary phase unless the profit share changes. Since productivity growth is exogenously given, real wage changes (due to unemployment changes) drive the cycle.

However, when one tries to take this model to the data, several assumptions do not find empirical support. While there is a sufficiently wide consensus about the unemployment effect on the *level* of the real wage (the so-called *real wage curve*), the evidence on the correlation between the *rate of change* of real wages and unemployment is rather weak[1]. In addition, while in the theoretical model effective demand does not play any role (since one additional unit of demand translates one-to-one in additional units of capital), in the actual world we observe a variable capital/output ratio[2]. In facts, if adjusting the capital stock requires time an increase of demand does not translate immediately into new equipment, but rather into an increase in the degree of capacity utilisation.

In this paper we propose a small macro-model that is inspired to Goodwinian ideas, but that accounts for a real wage curve (obtained as bargaining outcome) and cyclical variation of capacity utilisation (due to adjustment costs of the capital stock). The main similarity to Goodwin's original model is that we introduce a profit constraint on investment, due to some sort of credit rationing. In this way the Goodwinian feed-back can still operate: an increase in investment raises the existing capital stock and the associated labour productivity; as a consequence, workers' unions raise their wage claims. But this lowers firm's profitability and its investment activity. In this way we retain the possibility of distributional cycles. However, when capacity utilisation can vary and demand formation is taken into account, profit reduction cannot be taken for granted. A wage-share increase may accelerate consumption (due to workers' higher propensity to consume): this pushes demand and capacity utilisation up. If demand and utilisation in turn affect investment, this may delay the upper turning point of the cycle. Or it can even induce a cycle of the accelerator-multiplier type: an expansion (reduction) in demand, leading to over- (under-) utilisation of capacity, is self-reinforced by an increase (decrease) in investment, up to the point where investment is constrained (stimulated) by the existing distribution of income. Then a slowing down (an acceleration) of investment causes a turn in the business-cycle phase. In this way our model allows also for the possibility of demand

[1] Empirical evidence on the effects of unemployment on the level of real wage can be found in Blanchflower-Oswald 1994 and Layard-Nickell-Jackman 1991. Empirical validation of a Phillips curve for Italian real wages can be found in Fabiani and oths. 1997.

[2] See the data on 8 OECD countries reported in Flaschel-Groth 1995.

cycles in the Harrod-Kalecky tradition, based on the interplay of investment and aggregate demand (for a given distribution of the value added).

Our model is studied analytically to uncover its dynamical properties. The main results are: (*a*) the economy can have a multiplicity of stationary equilibria (with a higher o lower level of activity); (*b*) these equilibria can be either stable or unstable depending on parameter values; (*c*) there can exist attracting closed orbits (that is, limit cycles) around the equilibria; (*d*) more than one attractor (equilibria and orbits) can be simultaneously stable, so that the economy can be trapped in different asymptotic regimes depending on initial conditions; (*e*) permanent oscillations are caused either by the interplay between capacity and output, as originally proposed by Harrod (1939) and Kalecky (1971), or by a Goodwin (1969) mechanism where income distribution is crucial. The model is also simulated using parameter values estimated in a previous work: it turns out that the analytical result can be confirmed for reasonable values of the parameters.

The paper is organised as follows. In section 2 we present the theoretical model; in sections 3 and 4 some formal propositions regarding its dynamical properties are presented. Section 5 includes some simulations of the model under plausible (with reference to the Italian case) values of the parameters. Section 6 concludes.

## 2. *The theoretical model*

We assume the existence of two groups of agents, employees and employers. Only the latter are endowed with initial capital, and therefore are entitled to earn profits from production. Let us call them workers and capitalists. There is a finite number of agents in each group. Each capitalist runs a firm, and the workers in each firm are represented by a firm union. In each firm the local union bargains over the (real) wage with the corresponding capitalist. At the aggregate level, the resulting wage does not ensure full employment of the existing labour force. We assume a small open economy, and therefore capitalists are price takers in the commodity (and labour) market. The commodity price is normalised to unity, so that all variables are expressed in real terms.

There exist adjustment costs for capacity: production does not adjust immediately to existing demand, and therefore there is a cumulation/decumulation of inventories. This aspect is not explicitly modelled here, but it implies obviously a disequilibrium motion for output. A credit rationing mechanism is assumed: therefore, desired investment encounters an upper limit created by the availability of funds to the firm[3].

In addition to investment, the demand side is defined as follows: workers and capitalists exhibit a consumption function that is exponential in their income, and workers' income elasticity of consumption is higher than capitalists': therefore income distribution between wages and profits

[3] In Kiyotaki-Moore 1995 the budget constraint of each investing agent equates (optimal) consumption, investment and debt repayment with cash flow and new loans. Taking individual consumption proportional to individual labour income and keeping a constant financial position creates a positive relationship between (actual) investment and current profits, as argued in the text.

affects the aggregate propensity to consumption[4]. The exogenous components of demand are represented by public expenditure and the net trade balance.

The objective function of each capitalist is given by

$$(1)\qquad \max_{L,I} \int_t^{\infty} \exp(-Rs)\cdot\left[Y - W(1+T)L - \psi(I)\right]ds,\quad \psi' > 0,\ \psi'' > 0,\ \psi(0) = 0$$

under the following constraints

$$(2)\qquad Y = \left[L^{\sigma} + \chi K^{\sigma}\right]^{\frac{1}{\sigma}}$$

$$(3)\qquad \dot{K} = I - \delta K$$

$$(4)\qquad I \le \nu\left(\Pi\right) = \nu\left(Y - W(1+T)L\right),\ \nu' > 0$$

where $Y$ = firm production, $L$ = firm employment, $I$ = gross investment decision, $K$ = installed capacity, $R$ is capitalists' inter-temporal discount rate, $W$ = wage rate at the firm level, $T$ = tax wedge, $W(1+T)$ = total labour cost, $\psi$ = adjustment cost for the firm, $\chi$ = output-capital scale factor, $\delta$ = depreciation rate, $\Pi$ = gross profits of the firm, $\nu$ a concave increasing function. We assume $\sigma$ <1, so that r = 1/(s-1), the elasticity of substitution between capital and labour, is negative.

Equation (1) indicates the inter-temporal maximisation of profit under adjustment costs for the capital stock. Equation (2) describes a CES technology under constant returns to scale. Inequality (4) describes the existence of liquidity constraints for the investor: in addition to greater availability of funds, existing profitability constitutes a signal to the market of the potential solvability of the capitalist, and therefore more credit is offered whenever profits are high.

The optimal demand for labour that solves problem (1) is given by

$$(5)\qquad L^* = Y\cdot\left[W(1+T)\right]^{\rho}$$

or denoting logs of variables with small letters

$$(6)\qquad l = y + \rho(w+\tau)$$

where $\tau = \log(1+T)$[5]. The first order condition with respect to investment is derived from the Hamiltonian associated to problem (1). Let the adjustment cost function $\psi$ take the following exponential form: $\psi(I) = \frac{I^{\varphi+1}}{\varphi+1}$, with $\phi$>0, so that our hypotheses on its derivative are satisfied. Then the optimal investment decision satisfies

$$(7)\qquad I = (\hat{\lambda} - \hat{\mu})^{1/\varphi},\quad \hat{\lambda} = \lambda\cdot\exp(Rs),\ \ \hat{\mu} = \mu\cdot\exp(Rs)$$

[4] This assumption is common in most of the post-keynesian tradition, often invoking some sort of liquidity constrain. Analogous result can be replicated in a neo-classical model of growth with constant productivity of capital: see Bertola 1993 and 1994.

[5] When writing down the Lagrangian that incorporates the Hamiltonian and a Kuhn-Tucker multiplier associated to inequality (4) and deriving with respect to $L$, we are implicitly assuming that $\exp(-Rs) \neq \mu\nu'$, which is generic.

where $\lambda$ is the current-value costate variable and $\mu$ is a current-value Kuhn-Tucker multiplier associated with inequality (4). The dynamic paths of $\lambda$ and $\mu$ determine the optimal path of investment according to the associated first order conditions

$$(8) \qquad \begin{cases} \dot{\lambda} = (\delta + R)\lambda - \dfrac{\partial Y}{\partial K}\left(1 + \mu\nu'\right) \\ \mu\left(I - \nu\left(\Pi\right)\right) = 0, \ \ \mu \geq 0 \end{cases}$$

Without (perfect) information about the future paths of all the variables and without explicit modelling of the credit market it is impossible to derive analytical time functions for $\lambda$ and $\nu$ (and consequently for $\mu$). However something can still be said under specific assumptions. If static expectations are introduced, condition (7) can be rewritten as

$$(9) \qquad I = \left[\frac{\chi}{\delta + R} \cdot \left(\frac{Y}{K}\right)^{-\frac{1}{\rho}} \cdot (1 + \mu\nu') - \mu\right]^{\frac{1}{\varphi}} = \begin{cases} \left[\dfrac{\chi}{\delta + R} \cdot \left(\dfrac{Y}{K}\right)^{-\frac{1}{\rho}}\right]^{\frac{1}{\varphi}} & \text{for } \Pi \geq \overline{\Pi} \\ \nu(Y - W(1+T)L) & \text{for } \Pi \leq \overline{\Pi} \end{cases}$$

where $\overline{\Pi}$ satisfies $\overline{\Pi} = \nu^{-1}\left(\left[\frac{\chi}{\delta + R} \cdot \left(\frac{Y}{K}\right)^{-\frac{1}{\rho}}\right]^{\frac{1}{\varphi}}\right) = \nu^{-1}\left(\hat{I}\right)$, $\hat{I}$ being the unconstrained level of investment.

In other words, when the level of profitability is sufficiently high, investment is driven by the ratio between marginal productivity and replacement costs, as in the neo-classical version of the Tobin's *q*-theory; otherwise, labour costs can reduce current investment because they lower available funds. If we assume concavity for the $\nu$ function (i.e. the tightness of the constraint declines more than proportionately as profitability increases), a reasonable relationship between profits and investment (for any given level of unconstrained investment) can be represented as in figure 1.

[figure 1 about here]

A way to formalise this type of relationship is to postulate first a concave functional form for the $\nu$ function for $\Pi \leq \overline{\Pi}$[6]. Thus, for given $\hat{I}$, we assume

$$(10) \qquad \nu\left(\Pi\right) = \nu\left(\Pi; \hat{I}\right) = \left(\frac{\Pi}{\hat{I}}\right)^{\omega} \cdot \hat{I} = \Pi^{\omega} \cdot \hat{I}^{1-\omega} \quad , \quad 0 < \omega < 1, \ \Pi \leq \overline{\Pi}$$

Notice that $n\left(\hat{I}, \hat{I}\right) = \hat{I}$, so that $\overline{\Pi} = \hat{I}$. Now, expression (9) is a continuous, piece-wise differentiable function. In order to smooth the kink which takes place at $\Pi = \overline{\Pi} = \hat{I}$, we introduce a differentiable weight $\zeta$ that is almost zero for $\Pi > \overline{\Pi}$ and almost unity for $\Pi < \overline{\Pi}$. The full investment function can thus be approximated by means of a geometric mean of constrained and unconstrained investment:

$$(11) \qquad I = \nu\left(\Pi, \hat{I}\right)^{\zeta} \cdot \hat{I}^{1-\zeta} = \left[\Pi^{\omega} \cdot \hat{I}^{1-\omega}\right]^{\zeta} \cdot \hat{I}^{1-\zeta} = \hat{I} \cdot \left(\frac{\Pi}{\hat{I}}\right)^{\omega\zeta}$$

We define the following $\zeta$ function[7]

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6 For an alternative formulation see Appendix 1 of Checchi-Rampa (1996).

(12) $$\zeta = \frac{1}{2}\cdot\left(1+\frac{1-e^{\upsilon\cdot\log(\Pi/\hat{I})}}{1+e^{\upsilon\cdot\log(\Pi/\hat{I})}}\right) = \frac{1}{2}\cdot\left[1+f\left(\frac{\Pi}{\hat{I}}\right)\right]$$

Since $f(\cdot)\in(-1,1)$, it follows that $\zeta\in(0,1)$; *f*, and thus ζ, is a reverted logistic function, which takes on its high value for $\Pi<\overline{\Pi}$ and its low value for $\Pi>\overline{\Pi}$. The higher is υ, the quicker is the transition. The investment function represented by equation (11) is a weighed (geometric) average of a purely neo-classical investment function with adjustment cost for capital and a classical function (where in the extreme case ζ=ω=1 investment equals profits). The switch of regime occurs around $\overline{\Pi}$: below this threshold, income distribution affects accumulation; above it the driving force is represented by the diminishing marginal productivity of capital.

When we take logarithms of equation (11), taking (9) and (12) into account, we get

(13) $$\log(I)\equiv i=\frac{1}{\varphi}\cdot\left(-\frac{1}{\rho}(y-k)+\log\left(\frac{\chi}{\delta+R}\right)\right)+\omega\cdot\zeta(\Pi,I^*)\cdot\left\{\log(\Pi)-\frac{1}{\varphi}\cdot\left(-\frac{1}{\rho}(y-k)+\log\left(\frac{\chi}{\delta+R}\right)\right)\right\}$$
$$=\frac{1}{\varphi}\cdot\left(-\frac{1}{\rho}(y-k)+\log\left(\frac{\chi}{\delta+R}\right)\right)+\omega\cdot\zeta\cdot\left\{y+\log(1-\theta)-\frac{1}{\varphi}\cdot\left(-\frac{1}{\rho}(y-k)+\log\left(\frac{\chi}{\delta+R}\right)\right)\right\}$$

where we have used the definition $\Pi=Y(1-\theta)$, θ being the wage share[8].

Let us notice that actual investment is a positive function of current production (both because it raises marginal productivity of existing capital and it raises available funds) and a negative function of installed capacity (it lowers actual productivity) and of user cost of capital (it increases replacement costs). Labour costs have an ambiguous sign: whenever the elasticity of substitution is smaller than one in absolute value, an increase in the wage rate lowers actual investment. Vice versa in the alternative case[9].

Now, definition (3) can be approximated by expanding in Taylor's series around log(*I*/*K*) = log(δ) (the value of gross investment that keeps the capital stock constant) as follows

(14) $$\dot{k}=\frac{I}{K}-\delta\ =\exp\left(\log\left(\frac{I}{K}\right)\right)-\delta=\exp\left[\log(\delta)\right]+\exp\left[\log(\delta)\right]\cdot\left(\log\left(\frac{I}{K}\right)-\log(\delta)\right)+o\left(\log\left(\frac{I}{K}\right)-\log(\delta)\right)-\delta\cong$$
$$=\delta\cdot\left[i-k-\log(\delta)\right]$$

to obtain the dynamic equation of the capital stock

(15) $$\dot{k}=-\frac{\delta}{\varphi\rho}\cdot y+\frac{\delta(1-\varphi\rho)}{\varphi\rho}\cdot k+\delta\omega\zeta\cdot\left\{y+\log(1-\theta)-\frac{1}{\varphi}\cdot\left(-\frac{1}{\rho}(y-k)+\log\left(\frac{\chi}{\delta+R}\right)\right)\right\}+\frac{\delta}{\varphi}\cdot\log\left(\frac{\chi}{\delta+R}\right)-\delta\cdot\log(\delta)$$

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[7] Alternative formulations were available (for example $\zeta=1\Big/\left(1+e^{\upsilon\log\left(\Pi/\hat{I}\right)}\right)$); the present one was chosen for easiness of computation of its derivative. In fact observe that $\frac{dz}{d(\Pi/\hat{I})}=\frac{1}{2}\cdot\frac{df}{d(\Pi/\hat{I})}=-\frac{u}{4}\cdot\left[1-f^2\right]$.

[8] It is easy to see that, given (5), $\theta=\left(W(1+T)\right)^{1+\rho}$.

[9] Under a Cobb-Douglas technology the investment would be of course independent of the wage rate.

Moving to the wage determination, we assume that within each firm the capitalist and the local union bargain over the wage rate. Once the bargain has ended successfully, the capitalist hires the workers unilaterally (according to equation 5). The threat point for the union is the alternative wage outside the firm, while for the capitalist is the zero production/zero profits situation. In symbols the optimal wage solves the following program

$$(16) \qquad \max_{W} (W - V)^{\beta} (\Pi)$$

where $V = (1 - \eta U) W^{e} + \eta U B$ = alternative wage, $U$ = unemployment rate, $B$ = unemployment benefit, $W^e$ = wage expected outside the firm, $\beta$ = bargaining power of the union; $\eta$ is a positive parameter such that $\eta U$ can be interpreted as the probability of not finding a new job having been fired (thus one might expect that $\eta$ is close to unity). When replacing the definitions of $V$ and $\Pi$ into equation (16) and considering that in equilibrium $W^e = W$ (absence of leap-frogging), one gets

$$(17) \qquad W = \frac{\eta U}{\beta + \eta U} B + \frac{\beta}{\beta + \eta U} \cdot \frac{Y}{L(1+T)}$$

Equation (17) shows that in equilibrium the wage rate will be a weighted average of unemployment benefit and labour productivity. Given the fact that the latter surely exceeds the former, an increase in bargaining power for the union raises the optimal wage. As a consequence, the share of wages in income, $\theta$, will be positively correlated with union power and unemployment benefit, and negatively correlated with unemployment rate and productivity. Assume, with little loss of generality for subsequent analysis, that the unemployment benefit is equal to zero. Considering the definition of $U = 1 - \frac{L}{LF} = 1 - \frac{Y[W(1+T)]^{\rho}}{LF}$, where $LF$ indicates the total labour force, and applying the implicit function theorem, we get

$$(18) \qquad \theta = \frac{\beta}{\beta + \eta U} = \theta(\beta, y, LF), \ \ \theta_{\beta} > 0, \ \ \theta_{y} > 0, \ \ \theta_{LF} < 0$$

The wage share defined in (18) fluctuates between 1 (*full employment*) and $\frac{\beta}{\beta + \eta}$ (*absence of production*). If we want to use an explicit version of the previous function, and want to take into account the fact that we do not observe extreme values in the real world, we can restrict its fluctuations in some interval $[\underline{\theta}, \overline{\theta}]$. Recalling (5) and assuming an exponential relation between output and bargained wage, we may posit:

$$(19) \qquad \theta = \frac{\beta}{\beta + \eta U} = \frac{1}{1 + \frac{\eta}{\beta} \cdot \left[ \frac{LF - L}{LF} \right]} \cong \underline{\theta} + \frac{1}{\frac{1}{\overline{\theta} - \underline{\theta}} + \frac{\eta}{\beta} LF \cdot \left[ Y^{-\gamma} \right]} = \underline{\theta} + \frac{1}{\frac{1}{\overline{\theta} - \underline{\theta}} + \frac{\eta}{\beta} \cdot \exp[lf - \gamma \cdot y]}$$

The particular shape of equation (19) is depicted in figure 2. It shows that the wage share remains at boundary values for almost all the output range, and has its highest rate of increase (i.e., the derivative $d\theta / dy$ is at its maximum) at $y = lf / \gamma$.

[figure 2 about here]

It is now time to move to the production side. As we have already anticipated, production does not adjust immediately to effective demand because of the adjustment costs in capacity, and therefore inventories play a buffer role. The easiest way to model this assumption is to posit

$$(20) \qquad \dot{y} = \alpha\left[\log(Y^d) - y\right]$$

where $Y^d$ = effective demand. The effective demand is a weighed (geometric) average of three components: private investment, domestic consumption and exogenous demand $X$ (including external trade balance and public expenditure).

$$(21) \qquad Y^d = I^{\varepsilon} C^{\xi} X^{1-\varepsilon-\xi}$$

Domestic consumption is simply (log)proportional to available income. Given the existence of the two groups of agents with different propensities to consumption (respectively $c_w$ and $c_k$), the average propensity to consume depends on the functional distribution of income.

$$(22) \qquad C = Y^c = Y^{(\theta c_w + (1-\theta)c_k)}$$

Replacing equations (21) and (22) into (20) yields

$$(23) \qquad \dot{y} = \alpha\left[\varepsilon \cdot i + \xi \cdot \left[c_k + \theta \cdot \left(c_w - c_k\right)\right] \cdot y + (1-\varepsilon-\xi) \cdot x - y\right]$$

Using all previous definitions we are led to a system of two non-linear first-order differential equations of the first order (where $\zeta$ and $\theta$ are defined according to equations (12) and (19) respectively)

$$(24) \qquad \dot{y} = \alpha\left\{\frac{-\varepsilon\left[1-\omega\zeta\left(1+\varphi\rho\right)\right]}{\varphi\rho} + \xi\left[c_k + \theta \cdot \left(c_w - c_k\right)\right] - 1\right\} \cdot y + \alpha\left\{\frac{\varepsilon\left(1-\omega\zeta\right)}{\varphi\rho}\right\} \cdot k + \alpha\varepsilon\omega\zeta \cdot \log\left(1-\theta\right) + \alpha\left\{\frac{\varepsilon\left(1-\omega\zeta\right)}{\varphi}\right\} \cdot \log\left[\frac{\chi}{\delta+R}\right] + \alpha(1-\varepsilon-\xi) \cdot x$$

$$(25) \qquad \dot{k} = -\frac{\delta\left[1-\omega\zeta\left(1+\varphi\rho\right)\right]}{\varphi\rho} \cdot y + \frac{\delta\left(1-\varphi\rho-\omega\zeta\right)}{\varphi\rho} \cdot k + \delta\omega\zeta \cdot \log\left(1-\theta\right) + \frac{\delta\left(1-\omega\zeta\right)}{\varphi} \cdot \log\left[\frac{\chi}{\delta+R}\right] - \delta \cdot \log(\delta).$$

## 3. *Steady states and their stability*

The non-linear dynamics of our model is driven by two switching functions: the profit constraint on the investment decision (equation 11) and the distribution of income under wage bargaining (equation 19). This implies that any expansion produces two contrasting effects: on one side the output expansion leads to an increase in consumption expenditure (reinforced by higher wage claims of workers); on the other side the investment growth is constrained by the reduction in profitability (due to greater capital stock and/or higher wage costs). However when the wage share has reached its maximum value, consumption increases at the same rate as output does, while investment starts decelerating: this explain the turning point of recession. Similarly occurs for the turning point of recovery.

The reason of the cyclical behaviour can be as follows. When capital is low (with respect to the locus $\dot{k} = 0$ of its stationarity) investment demand drives production upwards; consumption is then stimulated, sustaining this motion. But investment is furtherly stimulated, since the marginal productivity of capital (roughly measured by the output-capital ratio) is increasing; in addition, also profits are increasing, so that possible liquidity constraints disappear initially: it follows that capital starts increasing more rapidly than output. At a certain point investment demand starts decelerating, due to this growth of the capital/output ratio: so output stops its growth and eventually crosses (vertically) its stationarity locus (its speed becomes negative) *while investment is still positive*: this means that the output-capital ratio starts decreasing, and the inducement to invest is weakened. Furthermore, when output is high a possible change in income distribution might turn the liquidity constraint on: the slow-down of investment becomes thus important and the growth of capital decreases until it becomes negative. It follows that (although consumption might still be high) aggregate demand is falling, driving output downwards. The economy undergoes a recession, during which the sequence of phenomena is reversed with respect to our previous description, until the lower turning point is reached.

The qualitative phase diagram of system (24)-(25) is depicted in figure 3.

[figure 3 about here]

We now want to be more precise about the analytics of our dynamical system. Since its behaviour depends on a large set of parameters, we restrict our discussion to a subset of it. To this end, we introduce the following mild assumptions

*Assumptions* (A): (A1) $0 > \rho > -1$

(A2) $0 < \varphi < 1$

(A3) $\frac{\delta}{1-\bar{\theta}} < \frac{\delta + R}{\chi} < 1$, which of course implies $\frac{\delta}{1-\underline{\theta}} < \frac{\delta + R}{\chi}$.

As regards assumption A1, we know (see note 8) that it is necessary if one wants the wage share to be increasing in the wage rate and hence, as stated in (18), increasing in output (decreasing in the unemployment rate). Assumption A2 is consistent with our hypotheses on the adjustment costs for capacity, specifying an upper bound for the degree of their convexity. Assumption A3 is not very strong and is easily verified.

In order to better understand the dynamical properties of system (24-25), let us start from the existence of stationary states. We are in a position to offer the following

Proposition P1. *If assumptions* A1-3 *are satisfied, the dynamical system* (25-26) *admits at least one and at most three stationary equilibria.*

Proof:

The proof can be undertaken in three steps:

(*a*) We leave to the reader to write down the equations for the loci $\dot{k} = 0$ and $\dot{y} = 0$. The weight $\zeta$ define by equation (13) and utilised in the investment function, changes sharply its value when the following equality is crossed:

$$k = (1 + \varphi\rho)y + \varphi\rho \cdot \log(1 - \theta) + \rho \cdot \log\left(\frac{\delta + R}{\chi}\right) \qquad (*)$$

It can be shown that under our assumptions A1-3 the graph of (*) lies always *below* the graph associated to the $\dot{k}=0$ locus, so that ζ is always equal to zero along this locus (*i.e.*, investment is always unconstrained in this region). In fact the value of the $\dot{k}=0$ locus is lower than that of (*) at *y*=0 and at $y = {}^{lf}\!/_{\gamma}$, and the slope of the former is greater than that of the latter. As a consequence the locus $\dot{k}=0$ is a straight line.

(*b*) If θ is constant, it can be proved that the loci $\dot{k}=0$ and $\dot{y}=0$ intersect only once and thus there exists a unique stationary equilibrium. In order to see this, compute the values of the two functions at $y=0$, verifying that the $\dot{k}=0$ locus has a higher intercept than the $\dot{y}=0$ one. Then let *c* be the aggregate propensity to consume: $c = c_k + \theta\left(c_w - c_k\right) < 1$. It follows that:

$$\text{(slope of } \dot{y}=0 \text{ with } \theta \text{ constant)} = 1+\varphi\rho\frac{1-\xi c}{\varepsilon} < 1+\varphi\rho\frac{1-\xi}{\varepsilon} < 1+\varphi\rho < \frac{1}{1-\varphi\rho} = \text{(slope of } \dot{k}=0\text{)},$$

where we have used the negativity of φρ, and the facts *c* < 1 and (1-ξ-ε) > 0, that is (1-ξ) > ε. Notice that the slope of the $\dot{y}=0$ locus, besides being possibly negative, decreases after it intersects the graph of (*). Since $\dot{k}=0$ is a linear increasing function, while $\dot{y}=0$ is piece-wise linear and convex, we conclude that (if θ is constant) the two loci have a unique intersection (see figures 4a and 4b).

(*c*) Now relax the constancy of the wage share: this means that, as γ*y* approaches *lf*, the wage share starts increasing, as stated by (19). Thus the $\dot{y}=0$ locus bends upwards. If γ is large enough, this locus becomes quite steep around the point $y = {}^{lf}\!/_{\gamma}$ while the slope rapidly decreases far from that point, where the wage share is constant (either low or high). As a consequence, we can have three intersections between the two loci $\dot{y}=0$ and $\dot{k}=0$, and hence three stationary equilibria[10]. This completes the proof.

The exact location of these equilibria is a very complicated function of all parameters, and we shall not insist on this point, focusing our interest in more qualitative properties. While the case of two equilibria in not generic (*i.e.* it is almost never observed when parameters vary), whether there are one or three equilibria depends crucially on the parameter γ appearing in expression (19) for the wage share. If γ is large enough a change in output around $y = fl/\gamma$ induces a strong shift in income distribution, and hence in consumption expenditure: thus a quite different level of capital is necessary to induce an investment expenditure which ensures market clearing. This non-linearity is the potential source of multiplicity of equilibria. In addition, the amount of diversity between workers' and capitalists' propensity to consume and the magnitude of the parameters ε and ξ is important for the number of equilibria: if (as stated in note 10) the condition $\frac{\varepsilon}{|\varphi\rho|} < 1-\xi c$ is satisfied far from $y = fl/\gamma$, non-uniqueness is more probable, given a large γ. When three equilibria do exist, they are ordered in ascending order of magnitude of both output and capital. Thus the economy can operate at different stationary regimes: high or low levels of demand, output and capacity. This is clearly in line with the role played by the effective demand in determining output and income distribution (quite a genuine Keynesian phenomenon).

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[10] Observe that the condition $\frac{\varepsilon}{|\varphi\rho|} < 1-\xi c$ (which implies a negative slope of the $\dot{y}=0$ locus) is sufficient but *not* necessary in order to obtain three stationary states: simply, it renders the three-fold intersection more robust (see figures 4c and 4d). Notice in addition that the above condition might obtain also for non-negligible values of ε, provided ξ is small. In other words, the *slope* of the $\dot{y}=0$ locus replicates the sigmoidal shape of the function $\theta\left(y\right)$ around the point $y = {}^{lf}\!/_{\gamma}$. Clearly, a large γ implies the possibility that the $\dot{y}=0$ locus intersects the $\dot{k}=0$ locus (at most) three times, and this can happen more easily if $\frac{\varepsilon}{|\varphi\rho|} < 1-\xi c$.

[figure 4 about here]

In order to better characterise the previous results, something must now be said about the stability of the stationary equilibria. As regards this point, one can prove the following:

Proposition P2. *Given the fact that investment is not profit-constrained at stationary states, then:*

P2.1. *If* $\alpha\varepsilon \approx \delta$ *and if only one stationary equilibrium exists, it can be locally stable or unstable according to whether, respectively,* $|fl - \gamma y^*|$ *is far from or close to zero (y* is the equilibrium level of output); in the former case an equilibrium is more (less) stable when* $fl - \gamma y^*$ *is positive (negative).*

P2.2. *If* $\alpha\varepsilon \approx \delta$ *and if there are three stationary equilibria, either all they are unstable (and this is more probable if* $\frac{\varepsilon}{|\varphi\rho|} > 1 - \xi c$*); or the middle one is unstable, while the remaining two are locally stable* (*this is more probable if* $\frac{\varepsilon}{|\varphi\rho|} < 1 - \xi c$).

P2.3. *If* $\alpha\varepsilon \neq \delta$ *in a substantial way, stationary equilibria are in general stable (unstable) if* $\alpha\varepsilon << \delta$ $(\alpha\varepsilon >> \delta)$. *However, if* $\alpha\varepsilon << \delta$ *and three equilibria exist, the middle one is unstable.*

Proof.

Recall that we have proved that investment is unconstrained at stationary states (point *a* in the proof of Proposition 1). This implies $\zeta = \frac{\partial\zeta}{\partial y} = \frac{\partial\zeta}{\partial k} \cong 0$ at stationary states. As a consequence one can show that the Jacobian computed at stationary states is the following matrix:

$$\mathbf{J} = \left[\begin{array}{c:c} -\frac{\alpha\varepsilon}{\varphi\rho} + \alpha\left\{\xi\left[c_k + \left(c_w - c_k\right)\left(\theta + \frac{d\theta}{dy}y\right) - 1\right]\right\} + o_{yy} & \frac{\alpha\varepsilon}{\varphi\rho} + o_{yk} \\ \hdashline -\frac{\delta}{\varphi\rho} + o_{ky} & \frac{\delta}{\varphi\rho} - \delta + o_{kk} \end{array}\right]$$

$$\cong \left[\begin{array}{c:c} -\frac{\alpha\varepsilon}{\varphi\rho} + \alpha\left\{\xi\left[c_k + \left(c_w - c_k\right)\left(\theta + \frac{d\theta}{dy}y\right) - 1\right]\right\} & \frac{\alpha\varepsilon}{\varphi\rho} \\ \hdashline -\frac{\delta}{\varphi\rho} & \frac{\delta}{\varphi\rho} - \delta \end{array}\right]$$

where the terms $o_{ij}$ depend on the value and derivatives of $\zeta$ at stationary states, and thus can be disregarded. The sign-structure of the Jacobian matrix is $\begin{bmatrix} \pm & - \\ + & - \end{bmatrix}$, so that no clear-cut conclusion can be drawn in general about the sign of its eigenvalues. However something can be said under the particular assumptions of Proposition 2.

(*i*) If a unique steady state exists and the equilibrium output is located far from threshold $lf/\gamma$, we know that $\partial\theta/\partial y \approx 0$. It follows that the top-left element of **J** is lower than $\alpha\varepsilon/|\varphi\rho|$, since the aggregate propensity to consume is lower than one. If in addition $\alpha\varepsilon \approx \delta$, then it is easy to check that $\mathrm{tr}(\mathbf{J}) < 0$ and $\det(\mathbf{J}) > 0$, which implies negative real parts of the eigenvalues, and hence local stability. Notice however that the equilibrium wage share (and hence the aggregate propensity to consume) is low when the equilibrium output is *below* $lf/\gamma$: in this case the trace of **J** is not only negative but lower than in the case $y > lf/\gamma$, and stability is stronger. On the other hand, if the equilibrium output level is located near the above threshold and $\alpha\varepsilon \approx \delta$, one has $\partial\theta/\partial y >> 0$ (provided that $\gamma$ is not negligible), and hence $\mathrm{tr}(\mathbf{J}) > 0$ and $\det(\mathbf{J}) < 0$. Thus one eigenvalue is positive and the equilibrium is unstable. This proves part P2.1

(*ii*) Now, assume that three equilibria exist. We know that the middle one has output located near the threshold $lf/\gamma$, so that the same arguments as above apply: this equilibrium is unstable. As regards the two equilibria in which output is displaced from $lf/\gamma$, observe that the condition $\frac{\varepsilon}{|\varphi\rho|} > 1 - \xi c$ is precisely the condition for positivity of the top-left element of *J*. If ε is large enough, then we have $\mathrm{tr}\left(\mathbf{J}\right) > 0$ and $\det\left(\mathbf{J}\right) < 0$, that is instability. The opposite holds when $\frac{\varepsilon}{|\varphi\rho|} < 1 - \xi c$. Part P2.2 is thus proved.

(*iii*) Part P2.3, finally, is proved in a completely similar manner.

As regards the initial claim in Proposition P2, it is indeed reasonable. It would seem odd that in a long-run steady equilibrium investment is permanently rationed. On the other hand, the particular assumption in parts P2.1 and P2.2 ($\alpha\varepsilon \approx \delta$) can be interpreted as saying that output or capacity shocks affect the two endogenous variables (directly, *via* the reaction of investment) in a similar way. This means that the capital-output ratio tends to be preserved, and thus there is no reason for further investment: from this point of view shocks tend to be "neutral", so that stability or instability must depend on consumption effects.

Now, recall that we have introduced the parameter γ as a sort of *total* elasticity of employment to output (see expression 19 above): thus the first part of Proposition 2 asserts that stationary states tend to be unstable when located near or above a well-defined level of "high employment". The reason is the following: under these circumstances any increase in output induces a substantial increase (or even acceleration) in demand (mainly consumption), since the wage share is high (or increasing): it follows that a further push to output is impressed by this excess demand. On the contrary, there can exist an "under-employment trap" (*i.e.* a stable stationary state with low demand and employment).

As regards the second part of Proposition P2, the conditions which strengthen stability of the "external" equilibria are the same as those for having a decreasing shape of the locus $\dot{y} = 0$, however with a zone in which it is sharply increasing (see figures 4a-b). This circumstance means that, as output increases, consumption does not increase substantially, so that in order to clear the market an increasing investment is required: but this can happen only if the capital-output ratio is decreasing. Thus exogenous shocks to demand are not self-sustaining, and the equilibria are stable. On the contrary, in the zone where the above-mentioned graph is increasing the wage share is rapidly rising with output, so stimulating strong reactions of consumption expenditure. From the logical point of view, this is equivalent to the interpretation of instability in the first part of the Proposition. One could say that this case is somehow similar to Harrodian instability: the Keynesian multiplier is large and highly pro-cyclical, so that the positive feedback between output and investment is strong.

The very last part of result P2.2 confirms the existence of two "traps" in which the economy might be locked persistently: one with high employment, output, capacity, wage share and hence consumption; and one with opposite properties. Only some exogenous shocks (in our model, the term *x*) can produce jumps between them.

The final result (P2.3) states that, all other parameters given, stability (instability) is induced by a low (high) value of ε, the elasticity of aggregate demand to investment: this means that the mechanism of investment, too, is responsible for the dynamical properties of our system. Provided that the wage share is poorly sensitive to output, a low elasticity of aggregate demand to investment confirms the traditional neo-classical result: the investment activity is self-stabilising. On the contrary, a large ε drives the economy away from stationary states: when output is increasing (decreasing) towards its steady value, investment is further stimulated (depressed), thus amplifying the motion. Notice also that a prompt reaction of producers to excess demand (*i.e.* a large value of α) induces instability. Finally, even if α is not large, and ε is low so that three equilibria can exist, the middle one is anyway unstable (for the same reasons mentioned before).

## 4. *Oscillations and limit cycles*

The previous results inform us only about *local* stability or instability of steady states. However in case of instability it might happen that our variables do *not* diverge to infinity, but approach some closed orbit around the repelling stationary equilibrium: the non-linearities are responsible for this possibility, which does not exist in linear systems. Indeed, as shown in the previous phase diagram (figure 3) in the case of a single stationary equilibrium, the direction of "arrows" (velocities out of equilibrium) confirms a rotating motion of variables in time.

In general, it is not possible to say whether this cycle is self-sustaining, or dumped, or explosive. But under some circumstance we can be more precise about the existence of self-sustaining cycles. In fact the following result holds true.

> Proposition P3. *If* $\underline{\theta} < \theta < \bar{\theta}$ *and* $y \neq lf/\gamma$ *at a stationary state, then for proper values of* α *and* δ *an increase in* ε *causes the system to undergo a Hopf bifurcation. Thus there exists a closed orbit around the stationary equilibrium. The period of this closed orbit is decreasing in* ε.

Proof.
We are assuming that the equilibrium level of output is displaced from the threshold $fl/\gamma$, but not too much so. This is because we want $dq/dy$ to be not very high, and at the same time we want the second-order derivatives (at least) of our non-linearities to be non-vanishing[11]. Our strategy will be to show that **J** has indeed complex eigenvalues under our assumptions, and that the real part of the eigenvalues (or equivalently the trace) of **J** crosses zero from below when ε increases.

We must first check for the existence of complex eigenvalues of **J**, which happens if $[\mathrm{tr}(\mathbf{J})]^2 - 4\det(\mathbf{J}) < 0$. If one defines $\mathbf{J} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$, the latter condition is equivalent to $(j_{11} - j_{22})^2 + 4 j_{12} j_{21} < 0$. In our case this means:

$$\left\{ -\frac{\alpha\varepsilon}{\varphi\rho} + \alpha\left( \xi\left[ c_k + (c_w - c_k)\left( \theta + \frac{d\theta}{dy} y \right) \right] - 1 \right) - \frac{\delta}{\varphi\rho} + \delta \right\}^2 < \frac{4\alpha\varepsilon\delta}{(\varphi\rho)^2}, \qquad \text{or:}$$

$$\left\{ \underbrace{-\alpha\varepsilon}_{<0} \underbrace{+\delta(1-\varphi\rho)}_{>0} \underbrace{+\alpha\varphi\rho\left( \xi\left[ c_k + (c_w - c_k)\left( \theta + \frac{d\theta}{dy} y \right) \right] - 1 \right)}_{>0} \right\}^2 < 4\alpha\varepsilon\delta \qquad (**)$$

---

[11] This is in fact a necessary condition for the existence of a Hopf bifurcation. Another condition is that for some parameter values the eigenvalues of the Jacobian matrix be complex with zero real part, and that in addition the real part be sensitive to changes in the parameters (see *e.g.* Guckenheimer-Holmes, 1983, chapter 3).

That the third term in the braces of the LHS of last expression is positive derives precisely from our current assumptions. In fact this can happen only if $dq/dy > 0$, given that the aggregate propensity to consume is lower than one; but the latter derivative must not be too large (it is "large" when $y = fl/\gamma$), otherwise it might be difficult to annihilate the trace of **J**, which is also our goal. Now it is apparent that we can increase ε enough for driving the LHS to zero from above; at the same time the RHS is raised by this operation, and then our inequality is satisfied: the eigenvalues are complex.

Notice finally that, given the representation $\mathbf{J} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$, in our case we have $j_{11} > 0$ and $j_{22} < 0$: it follows that $\mathrm{tr}(\mathbf{J}) < j_{11} - j_{22}$, where the latter term is the one appearing within braces on the LHS of previous inequality. Thus when the LHS is zero the trace of **J** is negative; by further increasing ε, the inequality keeps holding, while tr(**J**) rises. Provided that the values of the remaining parameters are properly chosen, we can thus find a value of ε for which tr(**J**)=0 and the previous inequality is still satisfied. Notice that this argument implies also that the real part of the eigenvalues of **J** are non-constant as ε varies.

Up to this point we have not considered the presence of the $o_{ij}$ terms in the Jacobian matrix of our system. This procedure is justified when one wants to evaluate approximately its eigenvalues. Indeed those terms (which depend crucially on the derivatives of the weight ζ) are very small and do not generically reverse the sign of the eigenvalues. However those quantities are *not* exactly null, nor are null their derivatives of any order (recall the exponential function for ζ). One can conclude that the second and third-order derivatives of equations 24 and 25 do not vanish.

All conditions are thus present to apply the Hopf Bifurcation Theorem (see note 11 above). We conclude that our dynamical system can exhibit closed orbits. Now recall that the period of a Hopf orbit is given by $2\pi / \mathrm{Im}(\tau)$, where τ is the complex eigenvalue of **J**. But it is also known that the imaginary part of this eigenvalue increases when our inequality (**) is strengthened. Thus the period of the orbit is decreasing in ε. This proves the proposition.

The preceding Proposition only ensures us about the existence of closed orbits: that is, we cannot say whether they are limit cycles, i.e. attractors. In facts the bifurcation of Proposition P3 can be supercritical (that is the closed orbit is an attractor) or it can be subcritical: in the latter case the trajectories starting from the interior of the orbit converge to the stationary state, and those starting outside the orbit diverge from the orbit itself.

That Hopf bifurcations in our dynamical system can be either supercritical or subcritical (depending on parameters' values) comes from the fact that the second- and third-order derivatives of (24) and (25) can be both positive and negative. The sign of these derivatives (which governs the type of bifurcation) depends on parameters' values and on the location of the stationary state. However, the exact calculation is a substantial undertaking in our case, since the so-called "normal form" of the dynamical system is not easily recoverable. We simply observe that the relevant signs change *generically* when parameters vary in some region of the parameter space. Thus stable limit cycles *can* exist. For these reasons in the next section we resort to numerical analysis to corroborate our claim about the possible existence of both super- and sub-critical bifurcations.

We must notice at this point that the existence of subcritical (*i.e.* unstable) Hopf closed orbits does not necessarily imply that important shocks to the endogenous variables, pushing them outside those orbits, lead to unbounded divergence of the system. In fact we put forward the following

Remark. *If there exist two stable equilibria which are associated with subcritical Hopf orbits, one can find parameter values such that these unstable orbits are surrounded by a third closed orbit which is an attractor of the dynamical system*

An analytical proof of this Remark failed us. However we offer the following argument in support of it.

Consider Figure 5, where we have drawn the two loci $\dot{k} = 0$ and $\dot{y} = 0$, together with the line λ, which is the graph of relation (*) appearing in the proof of Proposition 1, and the dashed area, which is the zone of the *y* variable where the wage share changes. Under the hypotheses of the present Remark, the system has two locally stable equilibria (E1 and E2 in figure 5), which are surrounded by unstable Hopf orbits (O1 and O2 respectively). This behaviour is described by the arrows drawn near those orbits in the figure. Our aim is to show that we can find a closed curve surrounding O1 and O2 where the non-linear dynamics points inwards. Coupled with the assumed instability of O1 and O2, this implies the existence and stability of an intermediate closed orbit O3.

Now recall that system (24-25) includes two peculiar non-linearities (the functions defining ζ and θ): if the parameters governing these non-linearities (υ and γ respectively) are large enough, the system is approximately linear in most part of the *y*-*k* plane, that is outside the λ line *and* the dashed area. It follows that, apart from the latter regions, the dynamics is equal to that of the linearisation of the system. Since the points E1 and E2 are locally stable by assumption, each of them is the globally stable equilibrium of a linear system (obviously, the linear systems at E1 and E2 are different). Thus one can find for E1 and E2 closed curves on which the linear dynamics points inwards: as an example we have drawn these curves as c1 and c2 respectively. Outside the dashed area and along the solid parts of c1 and c2, the actual behaviour of our non-linear system is identical to that of the corresponding linear part. In fact, it follows from the proof of Proposition 1 that ζ is surely constant above the $\dot{k} = 0$ locus, and thus this non-linearity is not at work. As regards the dashed region, simply choose c1 and c2 such that they can be connected by an horizontal segment like c5: we know from the analysis of the flow (see figure 3) that above the $\dot{k} = 0$ locus arrows point to "south-west", that is they point inwards with respect to c5.

As regards the part of the dynamical flow which lies below the $\dot{k} = 0$ locus, in this region the directional arrows point to "north-east". Thus we can draw the horizontal segment c4 (passing through the intersection of $\dot{k} = 0$ and c1), being sure that the arrows of the non-linear system point inwards on it. Finally, given again that arrows point to "north-east", one can find a line like c3 (more or less steep, not vertical) which connects c4 and c2 such that the non linear flow points inwards on it. Notice that, once crossed the $\dot{y} = 0$ locus from below, arrows by definition point to "north-west", and *a fortiori* point inwards with respect to c3. Thus the union of (the solid parts of) c1, c5, c2, c3 and c4 forms the closed curve we where looking for.

This completes our argument.

[figure 5 about here]

The previous discussion uncovers the possible existence of two different types of limit cycles. The first type (that we think of the *demand* type) takes place "near" a stationary state, and we may expect that along the orbit income distribution is not strongly perturbed; if it is, this has little accelerating effect on aggregate demand. When capacity has reached its highest (lowest) ratio to output, investment start decreasing (increasing), and the cycle repeats itself; however, the economy does not converge to the steady state, which is unstable due to locally significant multiplier effects (the derivative of the wage share is non-vanishing). The interplay between output and capital seems to be the most important mechanism at work here.

The second possible kind of limit cycle (of the *distributional* type) has a more sophisticated interpretation. As long as fluctuations are mild, the above interaction between capacity and output is self-stabilising (the stationary state is locally stable, and an unstable Hopf orbit exists around it); in addition, the wage share is sticky (either at low or at high levels). If, on the contrary, a larger shock is impressed on the system, employees suddenly raise (lower) their wage claims and consumption changes accordingly, driving all variables far away from the previous values. It takes a longer time to reverse this situation, and the reversal is not self-stabilising, since velocities are high at the turning points: the economy remains trapped in this permanent oscillation.

Finally, the behaviour of the economy is even richer if the latter limit cycle surrounds two repelling orbits, each of which in turn surrounds one locally stable stationary state. Now we have *three* "traps": a stable low-activity equilibrium; another equilibrium, again stable, with higher demand, capacity and consumption; and finally a wide orbital attractor. While bargaining relations are quiet in the former two, the latter is characterised by sudden changes in income distribution. Having in addition a longer oscillation period, this orbit bears some resemblance to "political" cycles: during a full cycle the bargaining power of workers and the distribution of income change, due to the changing climate of industrial relations.

## 5. *Model simulations*[12]

We now move to a numerical investigation of the dynamic properties of the model. In order to check whether this model has some resemblance to observed data, we will introduce parameters' values which are reasonable for the Italian economy in recent decades. Starting with the output equation (24), we set $\varepsilon = 0.20$ and $\xi = 0.5$ nearly in accordance with the observed average gross investment/GNP and private consumption/GNP ratios in the period 1951-1994 (constant price). The (log of) exogenous component of the aggregate demand (export, public expenditure and changes in inventories) is equal to about 15 in the decade 1985-94 (1990 prices), and we start from that value. It is not easy to find reliable estimates for the different propensities to consume, since neo-classical theory of saving is usually related to total wealth and not to current income. However, we refer to the propensity to save according to (personal) income distribution as a rough indicator of our (class) propensity to consume. Jappelli-Pagano (1994) report a median propensity to save of 0.17 for the lowest income quartile and 0.43 for the highest quartile.[13] Basing on these values, we set $c_k = 0.50$ and $c_w = 0.85$.

As far as the technology is concerned, in a previous work (Checchi-Rampa-Rampa 1996) we made use of a Cobb-Douglas technology, thus restricting our attention to the case of $\rho = -1$. This implies constant income shares, which we do not want to maintain here. Using the same data-set we can provide the empirical equivalent of equation (6) that allows us to identify the value of $\rho$. Using Johansen trace test, for the Italian private sector in the period 1951-93 we find the following (unique) cointegration relationship[14]

$$(l - y) - 0.69 \cdot (w + \tau)$$

Manera (1994) estimated for the manufacturing sector in the period 1954-1983 an elasticity of substitution of $-0.45$[15]. While we must accept a measure of $\rho$ lower than the unity (thus implying a pro-cyclical behaviour of the wage share), we are left with the problem of choosing a specific figure. This parameter often appears in combination with the $\varphi$ parameter, and we have found in simulation

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[12] Dynamic simulations are performed using the Runge-Kutta algorithm contained in the software DMC (Medio 1993).

[13] Data are referred to the 1987 Bank of Italy survey on household incomes, and report the saving propensity for individuals aged 50-54. See table 6.7 in Jappelli-Pagano 1994. We are obviously aware that these figures are heavily influenced by transitory components of personal income, but we could not find alternative evidence referred to Italy.

[14] The p-value associated to the existence of at least one cointegrating relationship is 0.00, while the corresponding value for the existence of at least two is 0.80. The test assumes absence of trend and the optimal lag structure in accordance with the Akaike criterion (1 lag).

[15] Rowthorn 1996 indicates that the median value for the capital/labour elasticity among several studies for OECD countries is -0.59.

that what is relevant is the product of the two. As regards the adjustment costs for the capacity, we notice that in order to obtain convex costs of adjustment is sufficient to have $\varphi>0$; if the adjustment costs are quadratic (as in most empirical work), this means a value of $\varphi = 1$. But this value might be excessive, because it would dominate the investment dynamics. Again Manera (1994) claims that "Total adjustment costs, expressed as a percentage of the investment asset price, are equal to 30.4% ..... Finally marginal adjustment costs amount to about 30% of the investment unitary cost"[16]. This given, we assume as starting points of the simulations the following values: $\rho = -0.50$ and $\varphi$ between 0.20 and 0.30. Notice also that from a theoretical point of view there should be a trade-off between $\alpha$ and $\varphi$, since a lower adjustment cost for capacity should imply a quicker output response to demand shocks.

The scale factor $\chi$ has been calibrated in order to obtain a reasonable value of the capital/output ratio at the equilibrium; since this ratio ranges between 2 and 4 in the period 1951-94, this means a maximum value for $\chi$ (maximum degree of capacity utilisation) of 0.50, and we chose this value. The speed of adjustment $\alpha$ is obviously hard to measure. In the empirical estimation of an equation similar to (20), assuming an error correction adjustment, we obtained a value of 0.37 for the transitory component, and a value of 0.17 for the error correction component. Given the fact our present adjustment is better interpreted as an error correction, we prefer the latter value. Since in addition this figure incorporated the impact of all demand components, we assume $\alpha = 0.50$ (which is roughly the ratio between the estimated coefficient, 0.18, and the incidence of the autonomous component on total effective demand, $1-\varepsilon-\xi=0.3$).

As regards the wage share dynamics, we notice that $\eta$ measures the probability of remaining unemployed given the rate of unemployment, and therefore must it be close to unity. The $\beta$ coefficient measures the relative bargaining power of the trade unions *vis a vis* the counterparts, and once again in the long run it must be not far from unity. For these reasons, we set $\eta/\beta = 1$ as a plausible ratio. The $\gamma$ parameter must be greater than 1 by definition; in the absence of further information, we have used alternative values for the $\gamma$ parameter, ranging from 2 to 5. The 1951-1994 average value for the (log of) the labour force is $lf = 10$. Looking at equation (19), let us remember that whenever we change $lf$ (which is scale dependent), we must change $\gamma$ accordingly. Regarding the range of oscillation of the wage share, the choice is rather arbitrary: since in the last forty years the gross wage share in the private sector of the economy oscillates between a minimum of 0.36 in 1951 and a maximum of 0.49 in 1975, we set $\underline{\theta} = 0.15$ and $\overline{\theta} = 0.70$ in order to allow our model economy to oscillate also far from a steady state, if the case.

Table 1 - Parameters set (in bold, the parameters that have been changed in each round)

| Parameters | Basic simulation | Limit cycle | one stable equilibrium with external attracting orbit | two stable equilibria with external attracting orbit |
|---|---|---|---|---|
| $\varepsilon$ | 0.20 | 0.20 | 0.20 | **0.17** |

[16] Manera 1994, p.160.

| | | | | |
|---|---|---|---|---|
| $\xi$ | 0.50 | 0.50 | **0.40** | **0.45** |
| $x$ | 15 | 15 | 15 | 15 |
| $c_w$ | 0.85 | 0.85 | 0.85 | 0.85 |
| $c_k$ | 0.50 | 0.50 | 0.50 | 0.50 |
| $\rho$ | - 0.50 | -0.50 | -0.50 | -0.50 |
| $\chi$ | 0.50 | 0.50 | 0.50 | 0.50 |
| $\alpha$ | 0.50 | 0.50 | 0.50 | 0.50 |
| $\eta/\beta$ | 1 | 1 | 1 | **0.50** |
| $\gamma$ | 2 | 2 | 2 | **4** |
| $lf$ | 10 | 10 | **22** | **40** |
| $\underline{\theta}$ | 0.15 | 0.15 | 0.15 | 0.15 |
| $\overline{\theta}$ | 0.70 | 0.70 | 0.70 | 0.70 |
| $R$ | 0.08 | 0.08 | 0.08 | 0.08 |
| $\delta$ | 0.06 | 0.06 | 0.06 | 0.06 |
| $\varphi$ | 0.30 | **0.20** | **0.205** | **0.20** |
| $\omega$ | 0.50 | 0.50 | 0.50 | 0.50 |
| $\upsilon$ | 5 | 5 | 5 | 5 |

The user cost of capital $R$ and the depreciation rate $\delta$ have been set respectively equal to 0.08 and 0.06. We are finally left with the specification of the $\zeta$ function. The $\omega$ parameter controls the degree of curvature of the $\nu$ function, and we have chosen an intermediate case by setting $\omega = 0.50$. Finally, if we want our expression (11) to approximate well equation (9), the $\upsilon$ parameter should be high enough and we set $\upsilon = 5$. We summarise the assumed parameters in Table 1. Considering these figures, we should keep in mind that our Assumptions *A*1-3 are always satisfied.

By observing figures 6a and 6b we easily recognise that this simulated model exhibits cyclical properties. This does not come as a surprise if we look at the Jacobian of the dynamical system. Using the initial set of parameters (first column of table 1) we observe a stable focus, where an entire cycle is clearly recognisable for the endogenous variables (figure 6a). When lowering the adjustment cost for capital, we are increasing the investment elasticity to expected and/or actual profits. The qualitative properties of the flow (directions and curvatures) are unaffected, but a stable limit cycle now emerges clearly (second column of table 1 and figure 6b).

[figure 6 about here]

When we increase the labour force (the *lf* parameter) we are reducing the possibility of income share fluctuations, since now the output is in a range where unemployment is sufficiently high to prevent significant wage increases when productivity rises. This explains why the unique equilibrium exhibits stability. However, it is quite easy to find a neighbourhood in the parameter space where an external closed orbit is still able to attract the economic system whenever the fluctuations of the output exceed a certain threshold (third column of table 1 and figure 7a). This is achieved by slightly decreasing the consumption impact on the effective demand.

We have already seen that it is possible to obtain up to three equilibria, two of which are stable (Proposition P.2.2). By further increasing the *lf* and $\gamma$ parameter and decreasing the $\eta/\beta$ parameter, we observe two locally stable equilibria surrounded by an attracting closed orbit (fourth column of table 1 and see figure 7b). What is the significance of these changes ? With these parameter changes we are strengthening the labour market reaction to output fluctuations, as if the trade union had become eager to contend the available profits. This produces two stationary regimes, which are robust whenever the system goes nearby. Otherwise the system keeps on fluctuating, even if the movement is irregular because of acceleration in the turning points.

[figure 7 about here]

If we accept the idea that our model captures cyclical behaviours at a sufficient extent, the question now becomes: what originates the cycle ? and which are its dynamical properties ? Besides what we have already said in previous section, we will now refer to the cycle simulated in figure 6b, once the model has already bifurcated to the a stable closed orbit. When observing figure 8, we notice that investment systematically anticipates output of about one fourth of the cycle length[17]. When we compare the simulated properties with those observed in empirical series, we observe that given the absence of shocks the simulated series exhibit a greater variance and of course higher persistence.[18] However our simulated model retains the property of greater variation of capital stock (with respect to investment), and the leading behaviour of investment. Analogous properties are observed in actual data for the capital stock, whereas in the case of the model this is transformed into a slightly lagging behaviour (instead of a slightly leading one).[19]

[figure 8 about here]

One can also use simulations to verify whether our model is able to produce the two different types of cycles (demand and distributional cycle, respectively). We can neglect the income distribution effect on consumption by simply assuming an identical propensity to consume for both groups of agents. Alternatively, we can produce a constant wage share on income by assuming that the alternative wage for the trade union is independent of the unemployment rate[20]. In this case we expect the model to produce essentially a *demand cycle* due to the interplay of demand and capacity. Indeed the system keeps on exhibiting an attractive limit cycle, and no sensible effect is detected on the cycle length from these changes. An increase in output raises the marginal productivity of capital and/or actual profitability, while an increase in capital stock lowers capital productivity. As soon as quantity adjustments are not instantaneous, there is room for expansionary cycles: output growth raises investment, which in its turn contributes to a further increase in demand. When the accumulated capacity is high enough to depress the investment, the cycle reverts downward and a cumulative process of disinvestment and reduced production starts again.

When we perform the alternative experiment of excluding the demand dynamics (for example setting the investment impact on demand equal to zero, namely $\varepsilon = 0$), we already know that the model is

[17] In Checchi-Rampa-Rampa 1996 we measured an average cycle length of 9.15 years for the period 1951-92. That measure was obtained from the eigenvalues of the linearised system estimated on Italian data.
[18] The AR(1) coefficient for output and capital are 0.41 and 0.94 (1956-1992 - yearly data referred to the private sector). The same coefficients in our simulations are 0.94 and 0.93.
[19] The correlation coefficients of $k_t$ with $y_{t-1}, y_t, y_{t+1}$ are 0.72, 0.60 and 0.35 respectively in actual data, whereas our simulations yield the values of 0.60, 0.85 and 0.98.
[20] In terms of parameter values, we consider the case of $\eta = 0$.

characterised by stable equilibria, and therefore limit cycles do not occur (see Proposition 2.2 and 2.3). Alternatively, one could make investment entirely dependent upon profits and not upon the neoclassical Tobin-*q*: in order to achieve this kind of formulation, we have forced our weight $\zeta$ to be always equal to one. Under this assumption, our model collapses to a sort of purely classical theory. The results of these simulations show that cycles tend to disappear in our model: equilibria become either stable, or unstable (the latter case emerges when one forces investment to become a large part of demand).

Therefore, the second type of limit cycle we observe (the one surrounding the stable foci - see figure 7b), that is characterised by significant changes in income distribution, is supported in an essential way by endogenous demand formation. In other words, our model is unable to replicate a purely classical type of cycle (as in the original Goodwin's model where capital/output ratio is constant), but it requires some sort of feedback from capital formation onto demand in order to produce oscillations. The outcome in this case is a cycle characterised by longer period (about 25 years), which can be accounted fro invoking sociological and political factors.[21]

## 5. *Conclusions*

In this paper we have analysed a model in which firms' accumulation activity, besides being subject to adjustment costs, is possibly profit-constrained. The latter aspect depends on a bargaining for wages with trade unions, where the bargaining power is affected by the level of activity and the unemployment rate. Profits depend also on the degree of substitutability between capital and labour. In addition, consumption demand is affected by income distribution, since employers and employees are characterised by different propensities to consume. Thus the interplay between effective demand, output, employment and wage bargaining might be expected to give rise to endogenous self-sustaining fluctuations.

This expectation is indeed confirmed by the analysis of a non-linear differential system which results from the previous hypotheses. It turns out that a wide range of dynamics is possible. First of all, more than one steady state can exist: under some circumstances, three equilibria do exist, one of which is unstable and two are stable. The latter are ordered in terms of output and employment (so, one would maintain, they could be Pareto-ranked). However, since both are stable, the economy can be trapped in the low-activity regime. Second convergence to the steady states takes always place in a non-monotonic, oscillating, fashion. One can also find unstable equilibria, especially whenever investment responds quickly to output (low adjustment costs). Given the assumed non-linearities, this implies the existence of stable closed (Hopf) orbits around equilibria. Third, even when steady states are locally stable, the parameters governing the bargaining process between firms and unions are so crucial that wage claims change sharply once passed a certain threshold of the unemployment rate. This change modifies significantly the average propensity to consume (and hence the

[21] This is the interpretation of the Goodwinian cycle proposed by Flaschel-Groth 1995.

composition of effective demand) during the cycle: it follows that the process of convergence to the steady state is interrupted and the economy sets in a wider orbital oscillation, which is a further attractor of the dynamical system. While the former type of cycle (Hopf) exhibits a shorter period, and is mainly due to the interplay between output and capacity; the latter orbit takes a longer time to be completed, and can be viewed as a sort of political-economic cycle, during which the wage-share changes greatly.

These results are confirmed by means of simulations which utilise parameter values close to some estimates related to the post-war Italian economy. In particular, it is shown that the basic simulation preserves some focal properties of observed macroeconomic series, while only slight perturbations of the parameters are needed in order to obtain closed orbits (both short-period and long-period ones). We suggest that the interpretation of real macroeconomic fluctuations can be better afforded by means of a model which includes *both* the capital-output interplay *and* the income distribution-demand mechanism.

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Figure 1 - A finance-constrained investment function

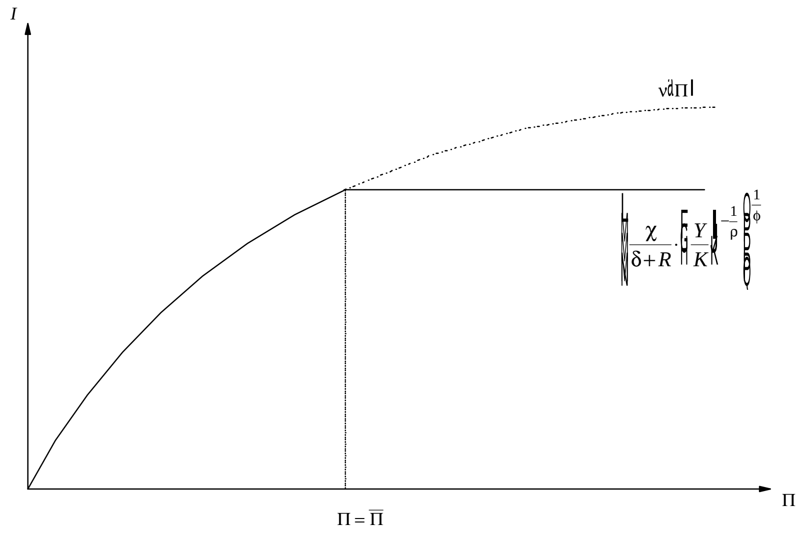



Figure 2 - The wage share θ behaviour

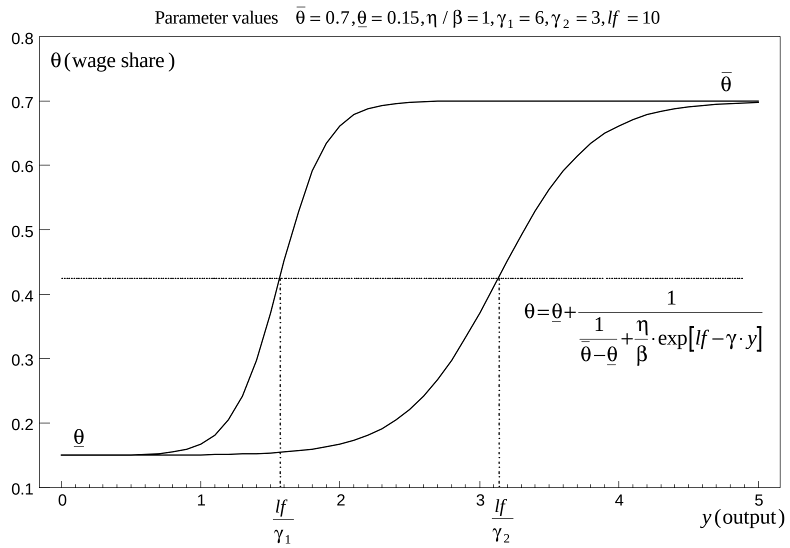



Figure 3 - Typical directions of motion

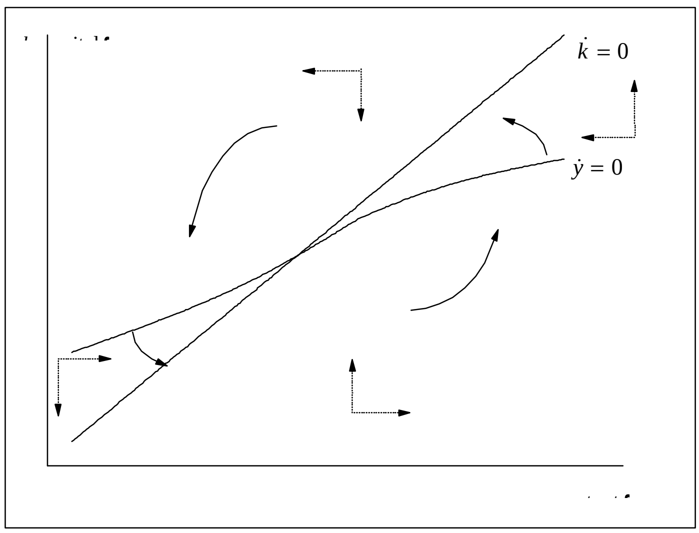
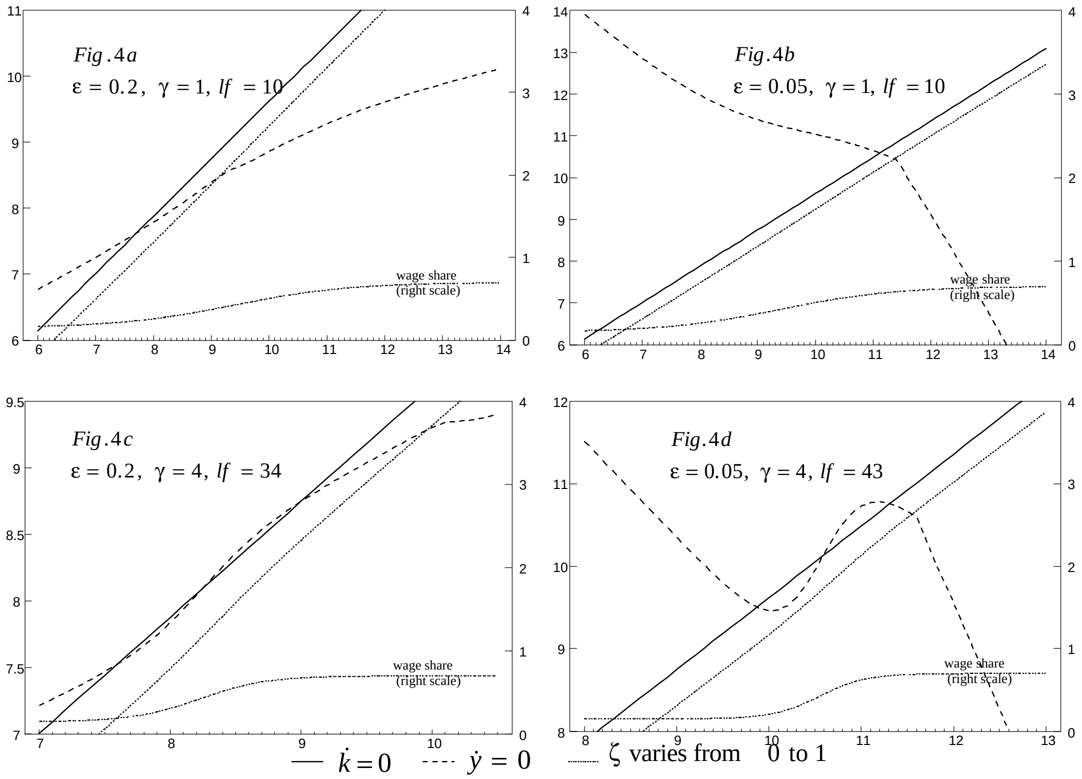


Fig.4 - Different configurations of phase diagram




Parameters values for the simulations: $\xi = 0.50$, $x = 15$, $c_w = 0.85$, $c_k = 0.50$, $\rho = -0.50$, $\chi = 0.50$, $\alpha = 0.50$, $\eta/\beta = 1$, $\underline{\theta} = 0.15$, $\overline{\theta} = 0.70$, $R = 0.08$, $\delta = 0.06$, $\varphi = 0.30$, $\omega = 0.50$, $\upsilon = 5$.

Fig. 5. The possible existence of a wider close orbit

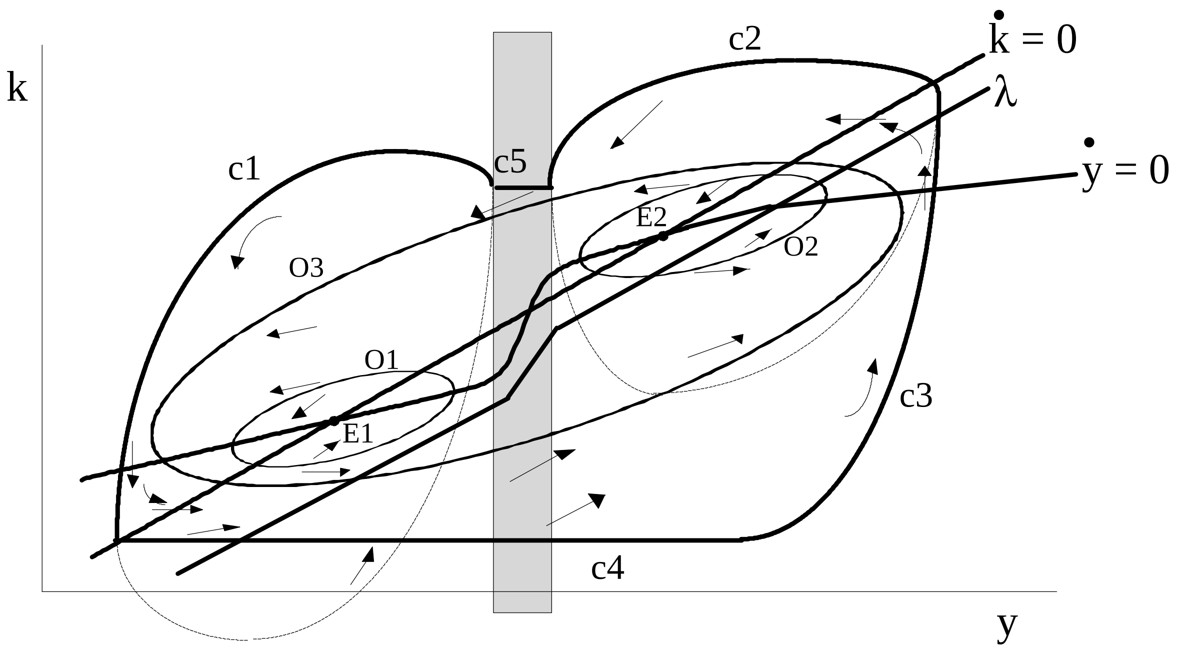


Fig.6a - Basic simulation (φ = 0.30)

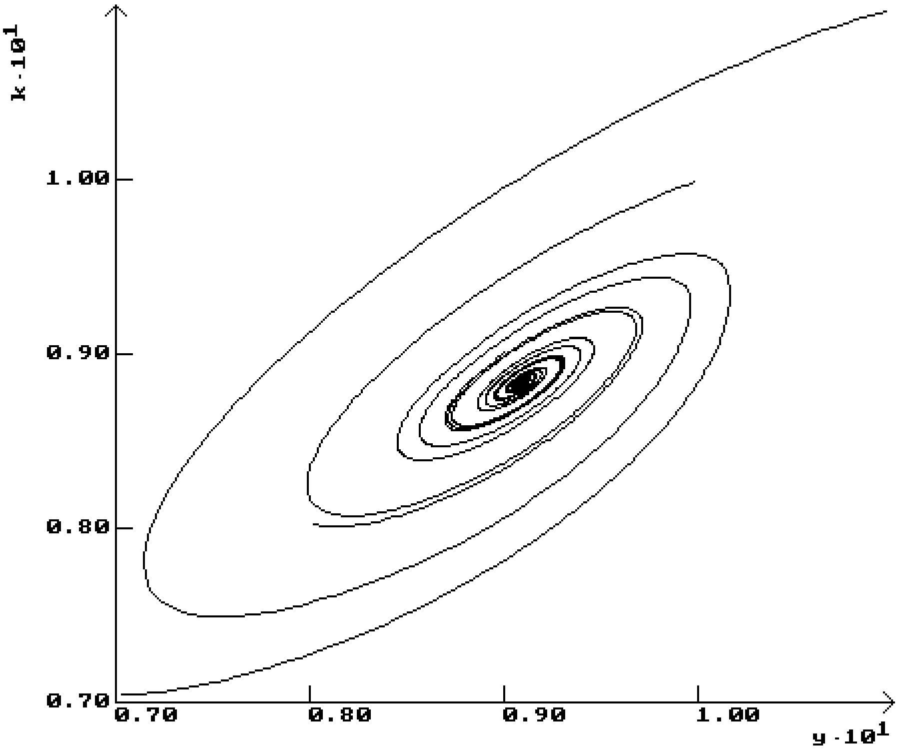



Fig.6b - A limit cycle (φ = 0.20)

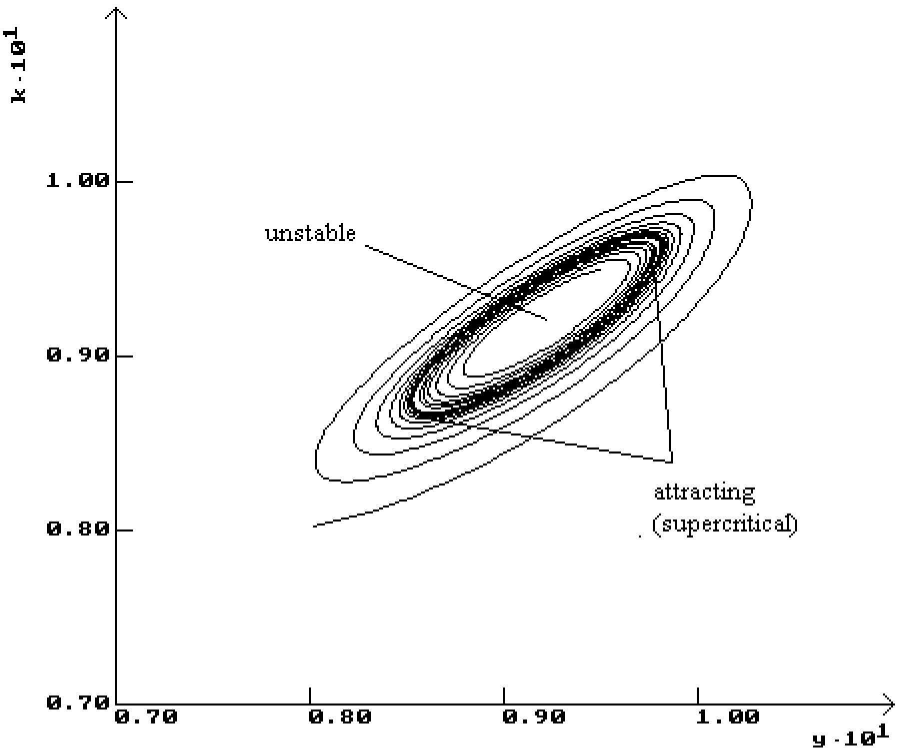


Fig.7a - A unique stable equilibrium surrounded by a repelling and an attracting orbit

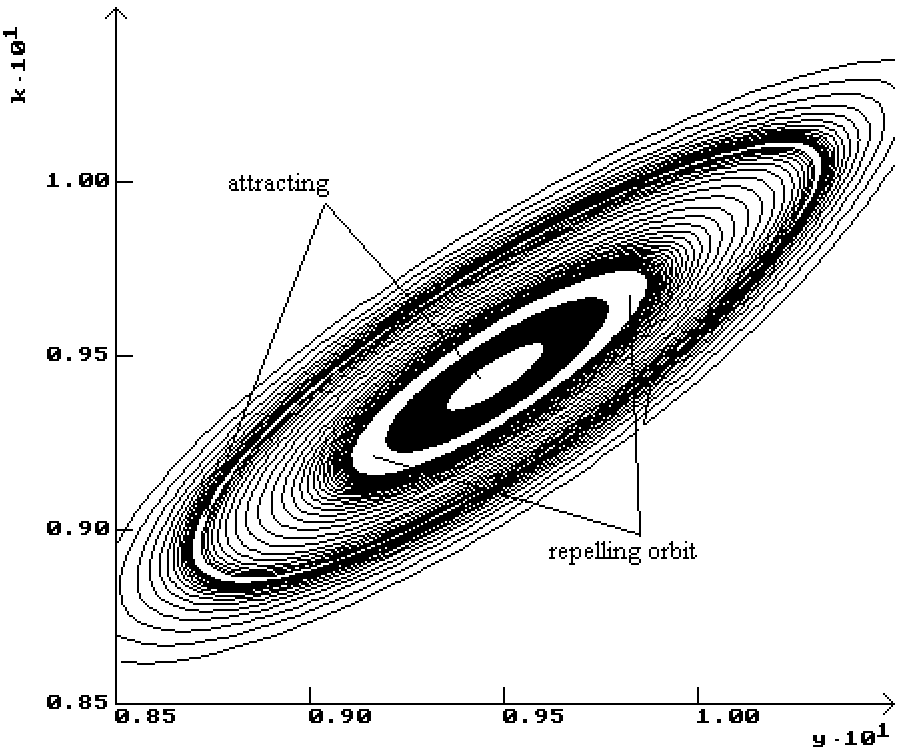



Fig.7b - Two stable equilibria surrounded by an attracting orbit

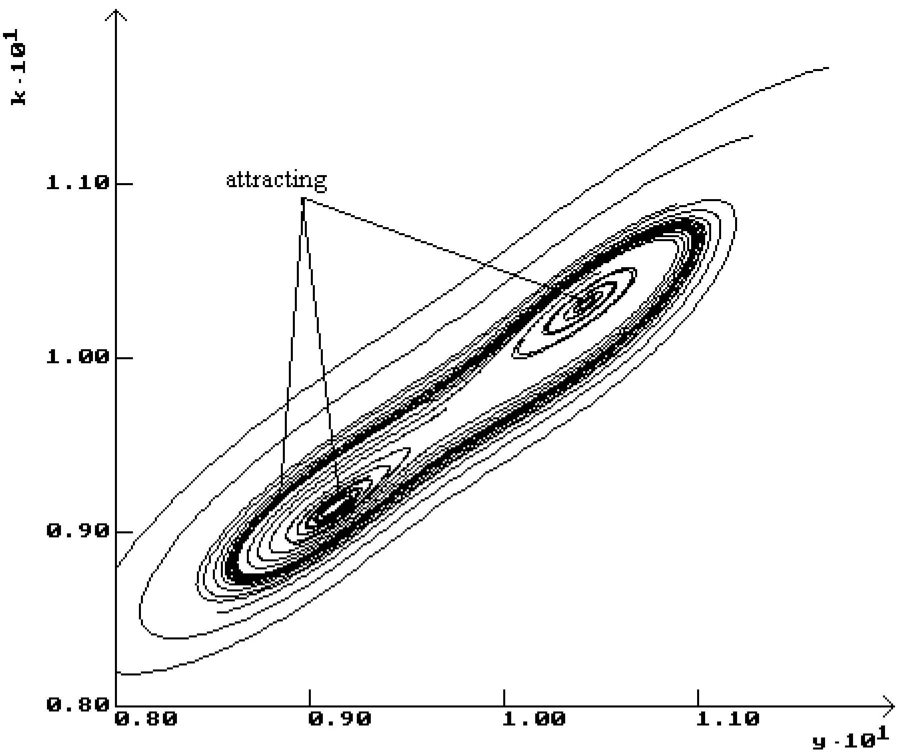
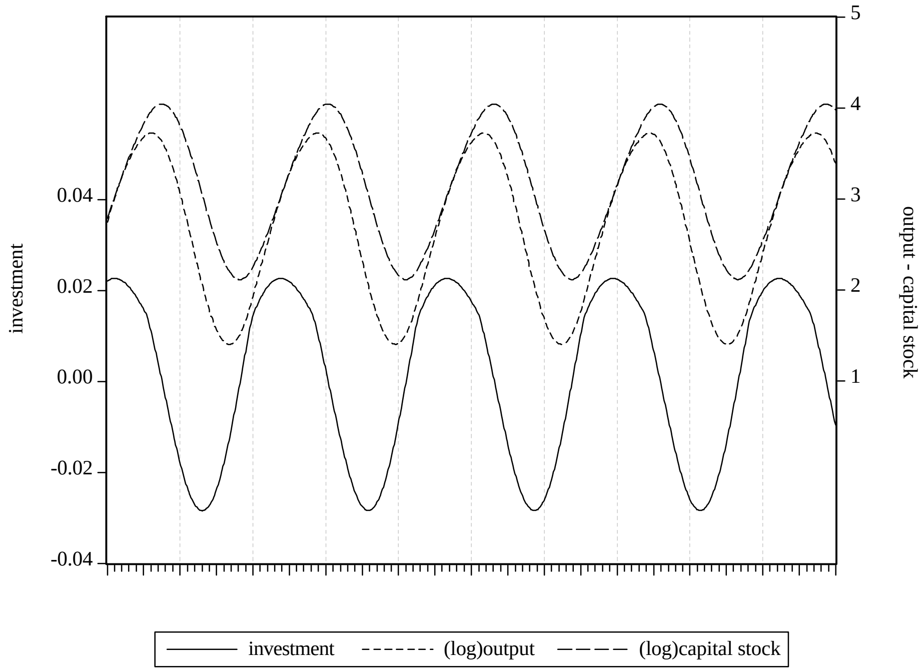


Fig.8 - Dynamics along the cycle



## For the referee

In the sequel we provide a more detailed sketch of the proofs of the three proposition in the text.

### Proof of Proposition P1

First of all, we must write down expressions for the equilibrium loci of equations 24 and 25:

(*i*) $$\dot{y}=0 \Leftrightarrow k = y+\rho\cdot\log\left(\frac{\delta+R}{\chi}\right)-\frac{\omega\zeta\varphi\rho}{1-\omega\zeta}\cdot\left[y+\log(1-\theta)\right]+\frac{\varphi\rho}{\varepsilon(1-\omega\zeta)}\cdot\left\{y-\xi\cdot\left[c_k+\theta\cdot(c_w-c_k)\right]y-(1-\varepsilon-\xi)\cdot x\right\}$$

(*ii*) $$\dot{k}=0 \Leftrightarrow k = \frac{1-\omega\zeta}{1-\omega\zeta-\varphi\rho}\cdot\left\{y+\rho\log\left(\frac{\delta+R}{\chi}\right)\right\}-\frac{\omega\zeta\varphi\rho}{1-\omega\zeta-\varphi\rho}\cdot\left[y+\log(1-\theta)\right]+\frac{\varphi\rho}{1-\omega\zeta-\varphi\rho}\cdot\log(\delta).$$

In addition to this, it can be checked that the weight $\zeta$ defined by expression (12) of the text, and utilised in the investment function, changes sharply its value when the following equality is crossed:

(*iii*) $$k=(1+\varphi\rho)y+\varphi\rho\cdot\log(1-\theta)+\rho\cdot\log\left(\frac{\delta+R}{\chi}\right);$$

when a ">" sign is present in (*iii*), $\zeta$ is roughly equal to zero (that is, investment is unconstrained by profits); when a "<" sign holds, $\zeta$ is almost equal to one. The higher is the parameter $\upsilon$, the more rapid is this transition. Recall that $\zeta=\zeta(y,k)$ and $\theta=\theta(y)$.

The proof requires some steps: in Step 1 it is shown that the graph of (*iii*) lies always below the graph of (*ii*) under our assumptions A1-3, so that $\zeta$ is always equal to zero along (*ii*) and thus the latter is a linear increasing function; in Step 2 we prove that, for $\theta$ constant (*i.e.* either $\eta/\beta = 0$ or $\gamma = 0$ or both) the loci (*i*) and (*ii*) intersect only once and thus there exists a unique stationary equilibrium; finally, in Step 3 we prove that allowing for a large parameter $\gamma$ causes the graph of (*i*) to exhibit a strong non-linearity, and thus to have possibly a three-fold intersection with (*ii*).

*Step* 1. Start at $y = 0$: this means that $\theta(0)=\underline{\theta}$, provided that *lf* (the log of labour-force) is positive, which we assume. Observe that, given Assumption A3, the vertical intercept of (*iii*) is positive. Now, suppose for a moment that $\zeta(0,k)$ is zero in expression (*ii*): if we can show that the vertical intercept of (*ii*) is larger than the one of (*iii*), then it follows that a ">" sign holds in (*iii*) at least around $y = 0$, and we are justified in assuming $\zeta = 0$. Indeed, call $\Gamma_2$ and $\Gamma_3$ the intercepts respectively of (*ii*) and (*iii*). Taking $y = 0$, $\zeta = 0$ and $\theta=\underline{\theta}$, one has:

$$\begin{aligned}\operatorname{sign}\{\Gamma_2-\Gamma_3\} &= \operatorname{sign}\left\{\frac{1}{1-\varphi\rho}\left[\rho\log\left(\frac{\delta+R}{\chi}\right)\right]+\frac{\varphi\rho}{1-\varphi\rho}\log(\delta)-\varphi\rho\log(1-\underline{\theta})-\rho\log\left(\frac{\delta+R}{\chi}\right)\right\}\\ &= \operatorname{sign}\left\{|\rho|\log\left(\frac{\delta+R}{\chi}\right)-\log(\delta)+(1-\varphi\rho)\log(1-\underline{\theta})\right\}\end{aligned}$$

where we have used the fact that $\varphi>0, \rho<0$ and hence $1-\varphi\rho>0$. Consider the argument of the last sign function:

$$\begin{aligned}&|\rho|\log\left(\frac{\delta+R}{\chi}\right)-\log(\delta)+(1-\varphi\rho)\log(1-\underline{\theta})\\ >\quad &\log\left(\frac{\delta+R}{\chi}\right)-\log(\delta)+\log(1-\underline{\theta})\\ =\quad &\log\left(\frac{\delta+R}{\chi}\right)-\log\left(\frac{\delta}{1-\underline{\theta}}\right)>0.\end{aligned}$$

We have used the following facts: $|\rho|<1$; $\delta+R<\chi$ (from A3); $\varphi\rho<0$; $|\varphi\rho|<1$ (from A1 and A2); $\underline{\theta}<1$; and finally A3 again. Thus it is confirmed that $\zeta$ is indeed zero at the vertical intercept of (*ii*).

Now, it is easy to see that the slope of (*iii*) is lower than the slope of (*ii*), if $\theta$ stays constant and $\zeta = 0$. In fact:

$$\text{slope of}\,(ii) \;=\; \frac{1}{1-\varphi\rho}=1+\frac{\varphi\rho}{1-\varphi\rho}>1+\varphi\rho \;=\; \text{slope of}\,(iii), \quad \text{since } \varphi\rho<0.$$

It follows that the vertical distance between the two loci increases. At a certain point (when $\gamma y$ approaches *lf*) $\theta$ starts rising and the graph of (*iii*) bends upwards: however, there is non crossing of the two curves. In fact if one writes down the general expression for their distance one finds that, in addition to the effect of *y* that is positive, this expression is identical to the expression we studied for the intercepts' distance, with $\bar{\theta}$ replacing $\underline{\theta}$; but assumption A3 implies that this distance is again positive. Thus $\zeta$ never becomes positive along the graph of (*ii*): this graph is a straight line with positive slope equal to 1/(1-φρ).

*Step* 2. Assume θ constant. Suppose $\zeta = 0$ at the intercept of (*i*); it is easily checked that this intercept is greater than that of (*ii*), and hence also the graph of (*i*) lies initially above (*iii*): ζ must actually be zero for $y = 0$. Now, let $c$ be the aggregate propensity to consume: $c = c_k + \theta(c_w - c_k) < 1$. The slope of (*i*) is clearly lower than the slope of (*ii*), in fact:

$$\text{(slope of } \dot{y}=0 \text{ with } \theta \text{ constant)} = 1+\varphi\rho\frac{1-\xi c}{\varepsilon} < 1+\varphi\rho\frac{1-\xi}{\varepsilon} < 1+\varphi\rho < \frac{1}{1-\varphi\rho} = \text{(slope of } \dot{k}=0\text{)},$$

where we have used the negativity of φρ, and the facts $c < 1$ and (1-ξ-ε) > 0, that is (1-ξ) > ε. Not only this, but when $y$ increases investment might become profit-constrained (we cannot exclude this case now), and the effect of $\zeta > 0$ is to further reduce the slope of (*i*). In fact this slope is $1+\varphi\rho\frac{[(1-\xi c)/\varepsilon]-\omega\zeta}{1-\omega\zeta}$, and, since $(1-\xi c)/\varepsilon > 1$, the term $\frac{[(1-\xi c)/\varepsilon]-\omega\zeta}{1-\omega\zeta}$ is increasing in ζ and hence that slope is decreasing in ζ (due to the negativity of ϕρ).

Observe that the slope of (*i*) could be negative: this happens if $\frac{\varepsilon}{|\varphi\rho|} < 1-\xi c$.

From this all it follows that (*ii*) is a linear monotonic function, while (*i*) is piece-wise linear and convex: we conclude that if θ is constant the loci (*i*) and (*ii*) have a unique intersection: see Figures A1*a* and A1*b*.

*Step* 3. See the text.

**Proof of Proposition P2**

Since we are now interested in *local* stability, let us write the derivatives of the non-linear terms of system (24-25).

*Wage share*. It follows from expression (19) of the text that: $\frac{1}{\frac{1}{\bar{\theta}-\underline{\theta}}+\frac{\eta}{\beta}\exp[lf-\gamma y]} = \theta-\underline{\theta}$. Hence:

$$\text{(D1)} \quad \frac{d\theta}{dy} = \gamma\cdot(\theta-\underline{\theta})^2\cdot\frac{\eta}{\beta}\cdot\exp[lf-\gamma y]$$
$$= \gamma\cdot(\theta-\underline{\theta})^2\cdot\left(\frac{1}{\theta-\underline{\theta}}-\frac{1}{\bar{\theta}-\underline{\theta}}\right)$$
$$= \gamma\cdot(\theta-\underline{\theta})\cdot\left(1-\frac{\theta-\underline{\theta}}{\bar{\theta}-\underline{\theta}}\right)$$
$$= \gamma\cdot(\theta-\underline{\theta})\cdot\frac{\bar{\theta}-\theta}{\bar{\theta}-\underline{\theta}} \geq 0$$

Observe that this derivative is zero on the frontier of the interval $[\bar{\theta},\underline{\theta}]$ and reaches the unique maximum at the middle point of it. From (19) it follows that as γ increases the above derivative stays near zero in most part of the interval.

*Log of profit share*. From (D1) it follows:

$$\text{(D2)} \quad \frac{d\log(1-\theta)}{dy} = -\frac{1}{1-\theta}\cdot\frac{d\theta}{dy}$$
$$= -\gamma\cdot\frac{\theta-\underline{\theta}}{1-\theta}\cdot\frac{\bar{\theta}-\theta}{\bar{\theta}-\underline{\theta}} \leq 0$$

This derivative reaches of course a minimum at the middle point of the interval $[\bar{\theta},\underline{\theta}]$.

*The ζ function*. We have defined $\zeta = \frac{1}{2}\left(1 + \frac{1 - e^{\upsilon \log(\Pi/\hat{\Pi})}}{1 + e^{\upsilon \log(\Pi/\hat{\Pi})}}\right) = \frac{1}{2}\left[1 + f\left(\log(\Pi/\hat{\Pi})\right)\right]$, where $f(z) \equiv \frac{1 - e^{\upsilon z}}{1 + e^{\upsilon z}}$ and $\upsilon >> 0$. In note 7 of the paper we observe that $\frac{d\zeta}{d \log(\Pi/\hat{\Pi})} = -\frac{\upsilon}{4} \cdot \left[1 - f^2\left(\log(\Pi/\hat{\Pi})\right)\right]$. Hence, from (9) and from the definition of profits, we obtain:

(D3) $$\frac{\partial \zeta}{\partial y} = \frac{d\zeta}{df} \cdot \frac{\partial f}{\partial y} = -\frac{\upsilon}{4} \cdot \left[1 - f^2\right] \cdot \left(1 + \frac{1}{\varphi\rho} + \frac{d \log(1-\theta)}{dy}\right) > 0, \quad \text{and} \quad \frac{\partial \zeta}{\partial k} = \frac{d\zeta}{df} \cdot \frac{\partial f}{\partial k} = -\frac{\upsilon}{4} \cdot \left[1 - f^2\right] \cdot \left(-\frac{1}{\varphi\rho}\right) < 0.$$

If $\upsilon$ is large $f$ remains most of the time near -1 or 1 (according to the sign of its argument), and thus the derivative of $\zeta$ is most of the time equal to zero.

*The investment function*. Given expression (13) of the text for investment, and our approximation (19) for the wage share, it follows that:

(D4) $$\frac{\partial i}{\partial y} = -\frac{1}{\varphi\rho} + \omega\zeta\left(1 + \frac{d \log(1-\theta)}{dy} + \frac{1}{\varphi\rho}\right) + \omega \frac{\partial \zeta}{\partial y}\left\{y + \log(1-\theta) - \frac{1}{\varphi}\left[\frac{1}{\rho}(k - y) + \log\left(\frac{\chi}{\delta + R}\right)\right]\right\};$$

(D5) $$\frac{\partial i}{\partial k} = \frac{1}{\varphi\rho}(1 - \omega\zeta) + \omega \frac{\partial \zeta}{\partial k}\left\{y + \log(1-\theta) - \frac{1}{\varphi}\left[\frac{1}{\rho}(k - y) + \log\left(\frac{\chi}{\delta + R}\right)\right]\right\}.$$

Collecting all the above material and taking account of the dynamical equations (24-25) of the text, it turns out that we can write the following expression for the Jacobian matrix of the system evaluated at any point (*y*,*k*):

(D6) $$\mathbf{J} = \left(\begin{array}{c|c} \alpha\varepsilon \frac{\partial i}{\partial y} + \alpha\left\{\xi\left[c_k + (c_w - c_k)\left(\theta + \frac{d\theta}{dy} y\right)\right] - 1\right\} & \alpha\varepsilon \frac{\partial i}{\partial k} \\ \hline \delta \frac{\partial i}{\partial y} & \delta \frac{\partial i}{\partial k} - \delta \end{array}\right).$$

Finally, recall that investment is unconstrained at stationary states. Given the previous arguments, this implies $\zeta = \frac{\partial \zeta}{\partial y} = \frac{\partial \zeta}{\partial k} = 0$. Thus the specialisation of D6 to stationary states gives the Jacobian matrix reported in the text.